Handout for MATH 036: Study Questions for "Bilateral Symmetry" (*Symmetry*, Chapter 1)

Alp Uzman

Last Revised: September 2, 2017

(i) How are beauty and symmetry related, according to Weyl? Observe that while the former is subjective the latter is objective. Could a deep study of symmetry lead to an objective theory of beauty (or aesthetics)? If yes, is this something to be worried about? Also keep in mind the quote from Dagobert on p. 16.

(ii) What is the typical scheme for all theoretical knowledge? Is there a typical scheme for other kinds of knowledge (e.g. common knowledge)?

(iii) Compare (platonic) idealism and realism. Which side is Weyl on? According to this how would you classify him, as a mathophysicist or a mathologist? Would this classification be legitimate?

(iv) Find all double-headed eagles used as symbols throughout history. Also research their meanings. Similarly, find all many-headed animals used as symbols throughout history (such as Cerberus from Greek mythology). Compare the meanings of many-headed animals to the meanings of double-headed eagles.

(v) On p. 13 Weyl makes a distinction between the oriental and the occidental: "For in contrast to the orient, occidental art, like life itself, is inclined to mitigate, to loosen, to modify, even to break strict symmetry.". Do you agree with Weyl? (Eastern) martial arts, Celtic knots, and knitting are important examples to keep in mind in this regard.

(vi) Research whether or not the houses of worship of all religions (that has a book or otherwise) have bilateral symmetry.

(vii) What does Leibniz mean when he uses the word "indiscernible" in a technical sense?

(viii) On p. 18 Weyl mentions one-to-one mappings (or transformations). These are precisely the one-to-one correspondences we talked about it class when discussing Halmos' article, except perhaps one-to-one mappings preserve additional structures. To give an example, recall the one-to-one correspondence between the fingers of the two hands. If we wanted to talk about a further structure on these "sets of fingers", we could refer to biology to recognize the uniqueness of the thumb among all other fingers. Indeed, the development of opposable thumbs is crucial to the mastery of utilizing tools. Then the one-to-one correspondence that sends the thumb of the left hand to the pinky finger of the right hand is not structure-preserving, though the "natural" one-to-one correspondence we talked about is structure-preserving. Using these definitions, find all one-to-one correspondences between fingers of hands. Find also all one-to-one mappings between fingers of hands. In particular compare the numbers of such correspondences.

(ix) Find examples of one-to-one, two-to-one, three-to-one, four-to-one, ..., many-to-one mappings. (Hint: Consider a rubber band wrapped around itself twice. Then cast its shadow onto a flat surface by using a light source directly above it. This would be 2 : 1, but from where (or what) to where (or what)?)

(x) What is an automorphism? Compare an automorphism to a metamorphism (possibly after defining the latter term appropriately, say, a la Kafka).

(xi) The left hemisphere of the brain controls the limbs on the right side of the body, while the right hemisphere of the brain controls the limbs on the left side of the body. Find an explanation regarding why such a crossing exists.

(xii) On p. 25 Weyl states that "a state of equilibrium is likely to be symmetric". What does he mean by this?

(xiii) How does Weyl justify that "contingency is an essential feature of the world" (on p. 26)? Also see the section on emergence of the "Cellular Automata" entry on Stanford Encyclopedia of Philosophy (https://plato.stanford.edu/entries/cellular-automata/).

(xiv) On p. 27 Weyl claims that "[t]he laws of nature do not determine uniquely the one world that actually exists, not even if one concedes that two worlds arising from each other by an automorphic transformation [...]". How does this relate to our discussion of chess, and consequently, to the paradox central to dynamics?

(xv) Recall the motto Halmos used in his article: "ontogeny recapitulates phylogeny". Considering the last sentence of the chapter, reevaluate this motto.

Handout for MATH 036: Study Questions for "Translatory, Rotational, And Related Symmetries" (*Symmetry*, Chapter 2)

Alp Uzman

Last Revised: September 20, 2017

(i) Can one identify a space with its identity mapping?

(ii) Let S_1, S_2 and T be three one-to-one mappings. If we denote "applying first S_1 and then S_2" by $S_2 \circ S_1$ (read "S_2 composed with S_1") and by T^{-1} the inverse of T, then on p. 42 Weyl states that $(S_1 \circ S_2)^{-1} = S_2^{-1} \circ S_1^{-1}$. For instance, to reverse "first wearing socks, then wearing shoes", you have to first take off the shoes, and then take of the socks, i.e.

$$\text{[first wearing socks, then wearing shoes]}^{-1} = (\text{[wearing shoes]} \circ \text{[wearing socks]})^{-1}$$
$$= \text{[wearing shoes]}^{-1} \circ \text{[wearing socks]} = \text{[taking shoes off]} \circ \text{[taking socks off]}.$$

Argue that the same order reversing would occur when we applied any number of one-to-one mappings one after the other, viz., if $S_1, \ldots, S_N$ are N one-to-one mappings, then

$$(S_N \circ \cdots \circ S_1)^{-1} = S_1^{-1} \circ \cdots \circ S_N^{-1}.$$

This phenomenon of order reversing is called **contravariance**.

(iii) What does Weyl mean by the word "space"?

(iv) On pp. 42-43 Weyl defines the group of automorphisms of a space, namely, if X is a space, then the set $\mathrm{Aut}\,(X)$ of all automorphisms of X satisfies the following three conditions:

(a) The identity mapping id_X of X is in $\mathrm{Aut}\,(X)$.
(b) If S is in $\mathrm{Aut}\,(X)$, then so is its inverse S^{-1}.
(c) If S and T are in $\mathrm{Aut}\,(X)$, then so is their composition $S \circ T$[1].

Check that the set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of all integer numbers has as its automorphism group the set of all mappings "add n", i.e.,

$$\mathrm{Aut}\,(\mathbb{Z}) = \{\ldots, \text{"add } -1\text{"}, \text{"add } 0\text{"}, \text{"add } 1\text{"}, \ldots\}.$$

Find $\mathrm{id}_{\mathbb{Z}}$, the inverse of "add n", and the composition "add n" $\circ$ "add m". Observe that the transformation "add 1" is nothing but shifting the space $\mathbb{Z}$ to the right by one unit.

(v) What is the difference between proper and improper congruences? (Hint: The modern terminology is "orientation-preserving" for "proper" and "orientation-reversing" for "improper").

(vi) Find spaces for which there are more than two categories of congruences. (Hint: What is the space we are working on when we apply proper and improper transformations?)

(vii) What is an infinite rapport?

[1] There is one more rule that Weyl does not list, which goes like this: if P, Q, R are in $\mathrm{Aut}\,(X)$, then $(P \circ Q) \circ R = P \circ (Q \circ R)$.

(viii) Research Leonardo da Vinci’s list of symmetry groups of the plane, which consists of all finite cyclic groups and all dihedral groups. (Hint: http://www-history.mcs.st-and.ac.uk/~john/geometry/Lectures/L8.html and p. 65 of Weyl.)

(ix) On p. 67 Weyl states that “[i]t seems that the origin of the magic power ascribed to these patterns lies in their startling incomplete symmetry—rotations without reflections”. Which patterns is he talking about? What is the automorphism group of said patterns? Research the different contexts said patterns were used throughout history.

(x) Research “phyllotaxis”. (Hint: If you attended the seminar I told you about in the beginning of the semester, it was about this.)

(xi) What is the Fibonacci sequence? Observe that this idea can be generalized quite easily, since in order to construct such a “nice” sequence all we need to do is to determine the general rule and two seeds from which we can start applying the formula. For instance, we could take $s_{n+2} = 2s_{n+1} + s_n$ as the general rule and $s_0 = -2, s_1 = 3$ as the seeds. Then we have the following sequence:

$$s_0 = -2, s_1 = 3, s_2 = 4, s_3 = 11, s_4 = 26, \ldots$$

Observe that we could have done this iteration not only in forward time, but also in backward time. Indeed, since $s_{n+2} = 2s_{n+1} + s_n$, we also know that $s_{n+2} - 2s_{n+1} = s_n$, or, shifting the indices by two, $s_n - 2s_{n-1} = s_{n-2}$. Thus in fact we also have the information on the history of this sequence:

$$\ldots, s_{-3} = 39, s_{-2} = -16, s_{-1} = 7, s_0 = -2, s_1 = 3, s_2 = 4, s_3 = 11, s_4 = 26, \ldots$$

In the same vain, find the past of the Fibonacci sequence.

(xii) List all the Platonic solids. Find the numbers of their vertices, edges, and faces. For instance, the cube is a Platonic solid. It has $V_{\text{cube}} = 8$ vertices, $E_{\text{cube}} = 12$ edges and $F_{\text{cube}} = 6$ faces. In particular, we have

$$V_{\text{cube}} - E_{\text{cube}} + F_{\text{cube}} = 8 - 12 + 6 = 2.$$

Show, by computing, that the same arithmetic operation would end up with a 2 for any Platonic solid.

Handout for MATH 036: Study Questions for “Ornamental Symmetry” (*Symmetry*, Chapter 3)

Alp Uzman

Last Revised: October 21, 2017

(i) Research the problem of “sphere packing”. (Hint: `https://youtu.be/ciM6wigZK0w`)

(ii) Explain why honeycombs have the hexagonal pattern.

(iii) On p. 89 Weyl claims that “[...] a hexagonal net covering the sphere is impossible owing to a fundamental formula of topology.” What is that fundamental formula. (Hint: Euler.)

(iv) What is the fundamental concept of metric geometry? Research what a distance function is. What is the relationship between a distance function and the fundamental concept of metric geometry?

Handout for MATH 036: Study Questions for "Crystals. The General Mathematical Idea Of Symmetry" (*Symmetry*, Chapter 4)

Alp Uzman

Last Revised: October 21, 2017

(i) What is the relation between symmetry and physics?

(ii) On p. 129 Weyl states that "[...] physics has revealed that an absolute standard length is built into the constitution of the atom, or rather into that of the elementary particles, in particular the electron with its definite charge and mass.". What is that absolute standard length? (Hint: see p. 133)

(iii) Consider the four dimensional continuum of space-time described on p. 131. What are the three dimensional layers of simultaneity and the one dimensional fibers of world-lines (or rest)? Draw a caricature that depicts them.

(iv) What does objectivity mean, in the context of general relativity?

(v) What is Klein's Erlangen program?

(vi) How does Weyl define an "algebraist"?

(vii) Research the life of Évariste Galois. What is the imaginary number *i*? What is the Galois group of the complex numbers over the real numbers?